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On the Distribution of Alternation Points in Uniform Polynomial Approximation of Entire Functions

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We consider the distribution of alternation points in best real polynomial approximation of a function $f \in C[-1, 1]$. For entire functions f we look for structural properties of f that will imply asymptotic equidistribution of the corresponding alternation points.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Suppose that $f \in C[-1, 1]$ is a real valued function which is not a polynomial and let

$$E_n = E_n(f) := \min_{p \in P_n} \|f - p\|_{[-1,1]} = \|f - p_n^*\|_{[-1,1]}, \qquad n \in \mathbb{N}_0,$$

denote the error of the best uniform approximation $p_n^* = p_n^*(f)$ to f in the set P_n of polynomials of degree at most n. By the Chebyshev equioscillation theorem there exist (not necessarily unique) alternation points $-1 \leqslant x_1^{(n)} < \cdots < x_{n+2}^{(n)} \leqslant 1$ such that for some $\delta_n \in \{-1, 1\}$, $n \in \mathbb{N}_0$,

$$(f - p_n^*)(x_j^{(n)}) = \delta_n (-1)^j E_n \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2.$$

In this note we will consider the asymptotic distribution of the corresponding unit counting measures ν_n, $n \in \mathbb{N}_0$, defined by

$$\nu_n(B) = \frac{\text{number of points } x_j^{(n)} \text{ in } B}{n+2} \qquad \text{for every set} \quad B \subset [-1, 1].$$

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From results of Kadec it follows that

THEOREM A (Kadec, cf. [6]). *There exists a subsequence $L = L(f)$ of $\mathbb{N}_0$ such that in the weak star topology we have*

$$\nu_n \xrightarrow{*} \mu_{[-1,1]} \qquad (n \in L), \tag{1}$$

where $\mu_{[-1,1]}$ denotes the equilibrium distribution of $[-1, 1]$.

Generalizations of this result and estimates on the discrepancy of ν_n and $\mu_{[-1,1]}$ have been given, for example, in [1, 3]. If we put $E_{n+1} = (1 - \varepsilon_n) E_n$, then it is known (cf. [9]) that the condition

$$\lim_{n \in L} \varepsilon_n^{1/n} = 1, \qquad \text{or equivalently,} \qquad \lim_{n \in L} \left(1 - \frac{E_{n+1}}{E_n}\right)^{1/n} = 1 \tag{2}$$

is sufficient (but not necessary) for (1).

In [9, 11] examples of entire functions f were constructed where (1) fails to hold for all n as $n \to \infty$. The question was raised in [9] by G. Lorentz: What structural properties of an entire function f ensure $\lim_{n \to \infty} \varepsilon_n^{1/n} = 1$, and thus the convergence of $(\nu_n)_n$ for all n as $n \to \infty$?

The following lemma gives a slightly generalized version of (2). For entire functions f it will imply a sufficient condition for (1) that depends on growth properties of f (cf. Theorem 2).

LEMMA 1. *Let L be a subsequence of $\mathbb{N}$ such that*

$$\lim_{n \in L} \left(1 - \frac{E_{[\alpha n]}}{E_n}\right)^{1/[\alpha n]} = 1 \qquad \textit{for all} \quad \alpha > 1.$$

Then (1) holds for L.

COROLLARY. *Suppose that*

$$\limsup_{n \in \mathbb{N}} E_n^{1/n} = 1/r \in (0, 1),$$

i.e., $\Gamma(r) = \{z \in \mathbb{C} \colon |z + (z^2 - 1)^{1/2}| = r\}$ is the largest ellipse with foci ± 1 such that f is holomorphic inside $\Gamma(r)$. Let L be a subsequence such that

$$\lim_{n \in L} E_n^{1/n} = 1/r.$$

Then, since $\limsup_{n \in L} E_{[\alpha n]}^{1/n} \leqslant 1/r^{\alpha} < 1/r$ for every $\alpha > 1$, Lemma 1 shows that (1) holds for L.

In the subsequent text we suppose that $f(z)=\sum_{j=0}^{\infty} a_j z^j$ is an entire function and define

$$\phi(r):=\max_{|z|=r} |f(z)| \qquad \text{for all} \quad r>0.$$

Let $M: [0, \infty) \to (0, \infty)$ be a continuous function that satisfies $\lim_{r\to\infty} M(r)/r^n=\infty$ for every $n\in\mathbb{N}$. Then the following properties hold.

LEMMA 2. *For every $n\in\mathbb{N}$ there exists some $r_n>0$ such that*

$$\frac{M(r_n)}{r_n^n}=\min_{r>0} \frac{M(r)}{r^n}=: \gamma_n.$$

Further, for every choice of r_n, $n\in\mathbb{N}$, we have

$$r_n \leqslant r_{n+1} \qquad \text{and} \qquad \lim_{n\to\infty} r_n=\infty.$$

In what follows we suppose that r_n, $n\in\mathbb{N}$, is an arbitrary choice of the numbers defined in Lemma 2 and that the function M gives a majorization of $|f|$ in the following sense:

$$\limsup_{n\in\mathbb{N}} \left(\frac{\phi(r_n)}{M(r_n)}\right)^{1/n} \leqslant 1. \tag{3}$$

Obviously, we may always choose $M=\phi$, but in many cases it might be easier to find some function M with the properties described above than to determine exact values for ϕ.

Remarks. (1) If f is of order $\rho\in(0, \infty)$ and type $\tau\in(0, \infty)$, a natural choice of M will be $M(r)=\exp(\tau r^\rho)$, such that

$$r_n=\left(\frac{n}{\tau\rho}\right)^{1/\rho} \qquad \text{and} \qquad \gamma_n=\left(\frac{\tau\rho e}{n}\right)^{n/\rho}.$$

Since, in this case, we have

$$\limsup_{r\to\infty} \frac{\log \phi(r)}{r^\rho}=\tau,$$

an elementary calculation shows that (3) is satisfied.

(2) For any entire function f of finite order ρ (without restrictions on the type) we may choose $M(r)=\exp(r^{\rho(r)})$, where $\rho(r)$ is a refined order for f (cf. [8, p. 30]).

By means of the γ_n, $n \in \mathbb{N}$, defined in Lemma 2 we can state a simple sufficient condition for (1) that corresponds to the corollary following Lemma 1.

THEOREM 1. *We have*

$$\limsup_{n \to \infty} \left(\frac{E_n}{\gamma_{n+1}} \right)^{1/n} \leqslant \frac{1}{2},$$

and if L is a subsequence of $\mathbb{N}$ *such that*

$$\lim_{n \in L} \left(\frac{E_n}{\gamma_{n+1}} \right)^{1/n} = \frac{1}{2},$$

then (1) *holds for L.*

Theorem 2 now gives a relation between the growth of f on certain radii r_n and the property (1) for certain subsequences L. The condition (4) is connected to the growth behavior of the majorant M (cf. Lemma 3), while (5) says that M should really match the behavior of $|f|$.

THEOREM 2. *Let L be a subsequence of* $\mathbb{N}$ *such that for some* $\delta > 0$ *we have*

$$\liminf_{n \in L} \frac{r_{[\alpha' n]}}{r_{[\alpha n]}} > 1 \qquad \textit{for all} \quad 1 \leqslant \alpha < \alpha' \leqslant 1 + \delta \tag{4}$$

and

$$\lim_{n \in L} \left(\frac{\phi(r_{[\alpha n]})}{M(r_{[\alpha n]})} \right)^{1/n} = 1 \qquad \textit{for all} \quad 1 \leqslant \alpha \leqslant 1 + \delta. \tag{5}$$

Then (1) *holds for L.*

We shall prove that (4) of Theorem 2 may be replaced by a condition on the growth of M. By the definition of r_n, we have for every $\beta > 0$

$$\left(\frac{M(\beta r_n)}{M(r_n)} \right)^{1/n} \geqslant \frac{\beta r_n}{r_n} = \beta.$$

Thus, Lemma 3 shows that a relatively modest growth of M at $r_{[\alpha n]}$, $n \in L$, implies (4).

LEMMA 3. *Let L be a subsequence of* $\mathbb{N}$ *such that for some* $\alpha \geqslant 1$ *we have*

$$\limsup_{n \in L} \left(\frac{M(\beta r_{[\alpha n]})}{M(r_{[\alpha n]})} \right)^{1/[\alpha n]} = \beta + o(\beta - 1) \qquad (\beta \to 1^+). \tag{6}$$

Then it follows that

$$\liminf_{n \in L} \frac{r_{[\alpha' n]}}{r_{[\alpha n]}} > 1 \qquad \textit{for all} \quad \alpha' > \alpha.$$

From Theorem 2 we immediately obtain

THEOREM 3. *Let f have finite order* $\rho \in (0, \infty)$ *and type* $\tau \in (0, \infty)$ *and suppose that f is of perfectly regular growth, i.e., that we have*

$$\lim_{r \to \infty} \frac{\log \phi(r)}{r^\rho} = \tau \qquad \textit{instead of} \quad \limsup_{r \to \infty} \frac{\log \phi(r)}{r^\rho} = \tau.$$

Then (1) *holds for* $L = \mathbb{N}$.

It is well known that f is of order $\rho \in (0, \infty)$ and type $\tau \in (0, \infty)$ if and only if

$$\limsup_{n \to \infty} n^{1/\rho} \, |a_n|^{1/n} = (\tau \rho e)^{1/\rho}.$$

By [10, p. 100], f is of perfectly regular growth if and only if there exists a subsequence $(n_k)_k$ of $\mathbb{N}$ such that $\lim_{k \to \infty} n_{k+1}/n_k = 1$ and

$$\lim_{k \to \infty} n_k^{1/\rho} \, |a_{n_k}|^{1/n_k} = (\tau \rho e)^{1/\rho}.$$

We note that functions of perfectly regular growth appear as solutions of linear differential equations with polynomial coefficients (cf. [5, pp. 204–208]).

Moreover, there are various results relating regularity conditions on the growth of f and the distribution of its zeros (cf., for example, [8, p. 88]).

2. PROOFS

Proof of Lemma 1. Suppose that (1) does not hold. Then there exists some $a \in (-1, 1]$ and $d > 0$ such that for some subsequence $(n_k)_k$ of L we have

$$|\nu_{n_k}([-1, a]) - \mu_{[-1,1]}([-1, a])| \geqslant d \qquad \text{for all} \quad k \in \mathbb{N}.$$

(1) We choose $\alpha > 1$ so close to 1 that $\alpha - 1 < d$ and define

$$e_k := \max_{n_k \leqslant \ell \leqslant [\alpha n_k] - 1} \varepsilon_\ell$$

and

$$m_k := \min\{ j \geqslant n_k : \varepsilon_j = e_k \text{ or } \varepsilon_j > 1/j \}.$$

We then have $n_k \leqslant m_k \leqslant [\alpha n_k] - 1$ and, since

$$\begin{aligned}
&\limsup_{k \to \infty} (1 - (1 - e_k)^{[\alpha n_k] - n_k})^{1/[\alpha n_k]} \\
&\quad \geqslant \limsup_{k \to \infty} \left(1 - \prod_{j = n_k}^{[\alpha n_k] - 1} (1 - \varepsilon_j) \right)^{1/[\alpha n_k]} \\
&\quad = \lim_{k \to \infty} \left(1 - \frac{E_{[\alpha n_k]}}{E_{n_k}} \right)^{1/[\alpha n_k]} = 1,
\end{aligned}$$

it follows by an elementary computation that $\lim_{k \to \infty} \varepsilon_{m_k}^{1/m_k} = 1$.

Further, we obtain that

$$1 \geqslant \lim_{k \to \infty} \left(\frac{E_{m_k}}{E_{n_k}} \right)^{1/m_k} = \lim_{k \to \infty} \left\{ \prod_{j = n_k}^{m_k - 1} (1 - \varepsilon_j) \right\}^{1/m_k} \geqslant \lim_{k \to \infty} \left(1 - \frac{1}{n_k} \right) = 1.$$

(2) The polynomial $p^*_{m_k + 1}(x) - p^*_{m_k}(x) = c_{m_k + 1} x^{m_k + 1} + \cdots$ satisfies

$$\begin{aligned}
|(p^*_{m_k + 1} - p^*_{m_k})(x_j^{(m_k)})| &\geqslant |(f - p^*_{m_k})(x_j^{(m_k)})| - |(f - p^*_{m_k + 1})(x_j^{(m_k)})| \\
&\geqslant E_{m_k} - E_{m_k + 1}
\end{aligned}$$

with alternating signs for all $1 \leqslant j \leqslant m_k + 2$. Since $\min_{p \in P_{m_k}} \| x^{m_k + 1} - p(x) \|_{[-1, 1]} = 1/2^{m_k}$, this implies (cf. [4, p. 77])

$$|c_{m_k + 1}| \geqslant 2^{m_k}(E_{m_k} - E_{m_k + 1}) = 2^{m_k} \varepsilon_{m_k} E_{m_k}.$$

The monic polynomial $(p^*_{m_k+1} - p^*_{n_k})(x)/c_{m_k+1} = x^{m_k+1} + \cdots$ therefore satisfies

$$\begin{aligned}
&\left\| \frac{(p^*_{m_k + 1} - p^*_{n_k})(x)}{c_{m_k + 1}} \right\|_{[-1, 1]} \\
&\quad \leqslant \frac{1}{2^{m_k}} \frac{\|(f - p^*_{n_k})(x)\|_{[-1, 1]} + \|(f - p^*_{m_k + 1})(x)\|_{[-1, 1]}}{\varepsilon_{m_k} E_{m_k}} \\
&\quad \leqslant \frac{1}{2^{m_k}} \frac{E_{n_k} + E_{m_k + 1}}{\varepsilon_{m_k} E_{m_k}} \leqslant \frac{1}{2^{m_k}} \frac{2E_{n_k}}{\varepsilon_{m_k} E_{m_k}},
\end{aligned}$$

and we obtain

$$\limsup_{k\to\infty} \left\| \frac{(p^*_{m_k+1}-p^*_{n_k})(x)}{c_{m_k+1}} \right\|^{1/m_k+1}_{[-1,1]} \leqslant \frac{1}{2}.$$

It follows by Theorem 2.1 in [2] that the zeros of $p^*_{m_k+1}-p^*_{n_k}$ are asymptotically equidistributed in $[-1,1]$. Thus, if λ_k denotes the number of zeros of $p^*_{m_k+1}-p^*_{n_k}$ in $\{z\colon \operatorname{Re}(z)\in[-1,a], \operatorname{Im}(z)\in[-1,1]\}$, we obtain

$$\lambda_k/(m_k+1)\to \mu_{[-1,1]}([-1,a]) \qquad (k\to\infty).$$

(3) Since

$$|(p^*_{m_k+1}-p^*_{n_k})(x_j^{(n_k)})| \geqslant E_{n_k}-E_{m_k+1}$$

with alternating signs for all $1\leqslant j\leqslant n_k+2$, there must be at least one zero of $p^*_{m_k+1}-p^*_{n_k}$ in each interval $(x_j^{(n_k)}, x_{j+1}^{(n_k)})$, $1\leqslant j\leqslant n_k+1$.

Therefore, if ξ_k denotes the number of $x_j^{(n_k)}$ in $[-1,a]$, it is not difficult to see that

$$\xi_k\leqslant \lambda_k+1 \qquad \text{and} \qquad \lambda_k\leqslant m_k-(n_k-\xi_k),$$

and thus

$$\frac{\lambda_k-(m_k-n_k)}{n_k+2}\leqslant v_{n_k}([-1,a])=\frac{\xi_k}{n_k+2}\leqslant\frac{\lambda_k+1}{n_k+2}.$$

An elementary computation yields

$$\begin{aligned}\alpha\mu_{[-1,1]}([-1,a])-(\alpha-1)\leqslant \liminf_{k\to\infty} v_{n_k}([-1,a])&\leqslant \limsup_{k\to\infty} v_{n_k}([-1,a])\\ &\leqslant \alpha\mu_{[-1,1]}([-1,a]),\end{aligned}$$

which by our choice of α, contradicts the assumption on $v_{n_k}([-1,a])$.

Proof of Lemma 2. Since

$$\lim_{r\to 0}\frac{M(r)}{r^n}=\lim_{r\to\infty}\frac{M(r)}{r^n}=\infty,$$

it is clear that r_n exists.

(1) Suppose that $r_{n+1}<r_n$. By the definition of r_{n+1} we have

$$\frac{M(r_{n+1})}{r_{n+1}^{n+1}}\leqslant\frac{M(r_n)}{r_n^{n+1}},$$

and thus

$$\frac{M(r_{n+1})}{r_{n+1}^n} \leqslant \frac{M(r_n)}{r_n^n} \frac{r_{n+1}}{r_n} < \frac{M(r_n)}{r_n^n},$$

which contradicts the definition of r_n.

(2) Suppose that there exists some $r>0$ such that $r_n \leqslant r$ for all $n \in \mathbb{N}$. Then, for every $s>r$,

$$M(s) \geqslant s^n \frac{M(r_n)}{r_n^n} \geqslant s^n \frac{\min_{t\in[0,r]} M(t)}{r^n} \qquad \text{for all} \quad n \in \mathbb{N},$$

which would imply that $M(s)=\infty$.

Proof of Lemma 3. If we suppose that $\liminf_{n\in L}(r_{[\alpha' n]}/r_{[\alpha n]})=1$ for some $\alpha'>\alpha$, then, since

$$\left(\frac{r_{[\alpha n]}}{r_{[\alpha' n]}}\right)^{[\alpha n]} \geqslant \frac{M(r_{[\alpha n]})}{M(r_{[\alpha' n]})} \geqslant \left(\frac{r_{[\alpha n]}}{r_{[\alpha' n]}}\right)^{[\alpha' n]},$$

there exists a subsequence $\tilde{L}$ of L such that

$$\lim_{n\in\tilde{L}} \left(\frac{M(r_{[\alpha n]})}{M(r_{[\alpha' n]})}\right)^{1/[\alpha n]} = 1.$$

Thus, for every $\beta>1$

$$\begin{aligned}
\limsup_{n\in L} \left(\frac{M(\beta r_{[\alpha n]})}{M(r_{[\alpha n]})}\right)^{1/[\alpha n]} &\geqslant \limsup_{n\in\tilde{L}} \left(\frac{M(\beta r_{[\alpha n]})}{M(r_{[\alpha n]})}\right)^{1/[\alpha n]} \\
&= \limsup_{n\in\tilde{L}} \left(\frac{M(\beta r_{[\alpha n]})}{M(r_{[\alpha' n]})}\right)^{1/[\alpha n]} \\
&\geqslant \limsup_{n\in\tilde{L}} \left(\frac{\beta r_{[\alpha n]}}{r_{[\alpha' n]}}\right)^{[\alpha' n]/[\alpha n]} = \beta^{\alpha'/\alpha},
\end{aligned}$$

which contradicts (6).

We state some simple inequalities which are needed in the proof of Theorem 1 and Theorem 2.

LEMMA 4. *For* $m \geqslant n$ *we have*

$$\left(\frac{r_m}{r_n}\right)^n \leqslant \frac{r_m^{m-1}}{r_n^n} \prod_{j=n+1}^{m-1} \frac{1}{r_j} \leqslant \frac{M(r_m)}{M(r_n)} \leqslant \frac{r_m^m}{r_n^{n+1}} \prod_{j=n+1}^{m-1} \frac{1}{r_j} \leqslant \left(\frac{r_m}{r_n}\right)^m,$$

and

$$\left(\frac{1}{r_m}\right)^{m-n} \leqslant \frac{\gamma_m}{\gamma_n} \leqslant \left(\frac{1}{r_n}\right)^{m-n}.$$

Proof of Lemma 4. By the definition of r_j and Lemma 2 it follows that

$$\frac{M(r_m)}{M(r_n)} = \prod_{j=n}^{m-1} \frac{M(r_{j+1})}{M(r_j)} \leqslant \prod_{j=n}^{m-1} \frac{r_{j+1}^{j+1}}{r_j^{j+1}} = \frac{r_m^m}{r_n^{n+1}} \prod_{j=n+1}^{m-1} \frac{1}{r_j} \leqslant \left(\frac{r_m}{r_n}\right)^m,$$

and

$$\frac{M(r_m)}{M(r_n)} = \prod_{j=n}^{m-1} \frac{M(r_{j+1})}{M(r_j)} \geqslant \prod_{j=n}^{m-1} \frac{r_{j+1}^j}{r_j^j} = \frac{r_m^{m-1}}{r_n^n} \prod_{j=n+1}^{m-1} \frac{1}{r_j} \geqslant \left(\frac{r_m}{r_n}\right)^n.$$

Since $\gamma_m/\gamma_n = (M(r_m)/M(r_n))(r_n^n/r_m^m)$, we obtain all estimates stated in the lemma.

Proof of Theorem 1. (1) Let $p_n \in P_n$ denote the polynomial that interpolates to f in the $n+1$ zeros of the Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos(x))/2^n = x^{n+1} + \cdots$. By [12, p. 50] we then have

$$E_n \leqslant \|(f-p_n)(x)\|_{[-1,1]} = \left\| T_{n+1}(x) \frac{1}{2\pi i} \int_{|\zeta| = r_{n+1}} \frac{f(\zeta)}{T_{n+1}(\zeta)} \frac{1}{\zeta - x} \, d\zeta \right\|_{[-1,1]}$$

$$\leqslant \frac{1}{2^n} \frac{M(r_{n+1})}{(r_{n+1}-1)^{n+1}} \frac{r_{n+1}}{r_{n+1}-1} = \frac{1}{2^n} \gamma_{n+1} \left(\frac{r_{n+1}}{r_{n+1}-1}\right)^{n+2}.$$

Since, by Lemma 2, $\lim_{n\to\infty} r_n = \infty$, this implies the first statement.

(2) Suppose that L is a subsequence such that

$$\lim_{n \in L} \left(\frac{E_n}{\gamma_{n+1}}\right)^{1/n} = \frac{1}{2}.$$

By the first part and Lemma 4 it follows that for every $\alpha > 1$

$$\limsup_{n \in L} \left(\frac{E_{[\alpha n]}}{\gamma_{[\alpha n]+1}}\right)^{1/[\alpha n]} \leqslant \frac{1}{2} \qquad \text{and} \qquad \frac{\gamma_{[\alpha n]+1}}{\gamma_{n+1}} \leqslant \left(\frac{1}{r_{n+1}}\right)^{[\alpha n]-n}.$$

An elementary calculation then shows that $\lim_{n \in L} E_{[\alpha n]}/E_n = 0$, and Lemma 1 yields (1) for the subsequence L.

Proof of Theorem 2. (1) Suppose that (1) does not hold. Then, by Lemma 1, there exists some $\tilde{\alpha} > 1$ such that

$$\liminf_{n \in L} \left(1 - \frac{E_{[\tilde{\alpha} n]}}{E_n}\right)^{1/[\tilde{\alpha} n]} < 1,$$

and thus for some subsequence $\tilde{L}$ of L

$$\lim_{n\in\tilde{L}} \frac{E_{[\tilde{\alpha}n]}}{E_n} = 1.$$

Without loss of generality we may assume that $\tilde{\alpha} \in (1, 1+\delta)$ and that $\tilde{L} = L$.

We fix some $\alpha \in (1, \tilde{\alpha})$ and obtain by Theorem 1 and Lemma 4

$$\begin{aligned}\limsup_{n\in L} \left(\frac{E_n}{\gamma_{[\alpha n]}}\right)^{1/[\alpha n]} &= \limsup_{n\in L} \left(\frac{E_{[\tilde{\alpha}n]}}{\gamma_{[\tilde{\alpha}n]}} \frac{\gamma_{[\tilde{\alpha}n]}}{\gamma_{[\alpha n]}}\right)^{1/[\alpha n]} \\ &\leqslant \left(\frac{1}{2}\right)^{\tilde{\alpha}/\alpha} \limsup_{n\in L} \left(\frac{\gamma_{[\tilde{\alpha}n]}}{\gamma_{[\alpha n]}}\right)^{1/[\alpha n]} \\ &\leqslant \left(\frac{1}{2}\right)^{\tilde{\alpha}/\alpha} \limsup_{n\in L} \left(\frac{1}{r_{[\alpha n]}^{[\tilde{\alpha}n]-[\alpha n]}}\right)^{1/[\alpha n]} = 0.\end{aligned}$$

(2) It is well known that $\lim_{n\to\infty} p_n^*(z) = f(z)$, and thus

$$f(z) = p_0^* + \sum_{j=1}^{\infty} p_j^*(z) - p_{j-1}^*(z),$$

locally uniformly for all $z \in \mathbb{C}$.

For every $n \in \mathbb{N}$ we put $R_n := r_n + (r_n^2 + 1)^{1/2}$. Then, $\{z: |z| \leqslant r_n\}$ is contained inside the ellipse $\{z: |z + (z^2-1)^{1/2}| = R_n\}$. Since $\lim_{n\to\infty} r_n = \infty$, we have

$$R_n = 2r_n(1 + d_n^{(1)}), \qquad \text{where} \qquad \lim_{n\to\infty} d_n^{(1)} = 0.$$

We fix some $\alpha' \in (1, \alpha)$. For all $|z| = r_{[\alpha' n]}$ the Bernstein–Walsh Lemma (cf. 12, p. 77]) yields

$$\begin{aligned}|f(z)| = \left|p_0^* + \sum_{j=1}^{\infty} (p_j^* - p_{j-1}^*)(z)\right| &\leqslant |p_0^*| + \sum_{j=1}^{\infty} \|p_j^* - p_{j-1}^*\|_{[-1,1]} R_{[\alpha' n]}^j \\ &\leqslant |p_0^*| + \sum_{j=1}^{\infty} (\|f - p_{j-1}^*\|_{[-1,1]} + \|f - p_j^*\|_{[-1,1]}) R_{[\alpha' n]}^j \\ &\leqslant |p_0^*| + \sum_{j=1}^{\infty} 2E_{j-1} R_{[\alpha' n]}^j.\end{aligned}$$

By Theorem 1 we have

$$E_{j-1} \leqslant \frac{\gamma_j}{2^j}(1+d_j^{(2)})^j, \qquad \text{where} \quad \lim_{j\to\infty} d_j^{(2)}=0.$$

To estimate $|f(z)|$ for $|z|=r_{[\alpha' n]}$ we split the series $\sum_{j=1}^{\infty} E_{j-1} R_{[\alpha' n]}^j$ into three parts

$$\sum_{j=1}^{\infty} E_{j-1} R_{[\alpha' n]}^j = \sum_{j=1}^{n} \cdots + \sum_{j=n+1}^{[\alpha n]} \cdots + \sum_{j=[\alpha n]+1}^{\infty} \cdots = S_{1,n}+S_{2,n}+S_{3,n}.$$

(a) For every $1 \leqslant j \leqslant n$ we have by Lemma 4

$$\begin{aligned}
E_{j-1} R_{[\alpha' n]}^j &\leqslant \frac{1}{2^j}\gamma_j R_{[\alpha' n]}^j (1+d_j^{(2)})^j \leqslant \gamma_j r_{[\alpha'_n]}^j (1+d_j^{(2)})^j (1+d_{[\alpha' n]}^{(1)})^j \\
&= M(r_{[\alpha' n]}) \left\{ \frac{M(r_j)}{M(r_{[\alpha' n]})} \frac{r_{[\alpha' n]}^j}{r_j^j} \right\} (1+d_j^{(2)})^j (1+d_{[\alpha' n]}^{(1)})^j \\
&\leqslant M(r_{[\alpha' n]}) \left\{ \frac{r_j^j}{r_{[\alpha' n]}^{[\alpha' n]-1}} \left(\prod_{k=j+1}^{[\alpha' n]-1} r_k \right) \frac{r_{[\alpha' n]}^j}{r_j^j} \right\} (1+d_j^{(2)})^j (1+d_{[\alpha' n]}^{(1)})^j \\
&= M(r_{[\alpha' n]}) \left\{ \frac{1}{r_{[\alpha' n]}^{[\alpha' n]-1-j}} \left(\prod_{k=j+1}^{[\alpha' n]-1} r_k \right) \right\} (1+d_j^{(2)})^j (1+d_{[\alpha' n]}^{(1)})^j.
\end{aligned}$$

We choose some arbitrary $\alpha'' \in (1, \alpha')$. Lemma 2 yields

$$\begin{aligned}
\{\cdots\} &= \frac{1}{r_{[\alpha' n]}^{[\alpha'' n]-1-j}} \left(\prod_{k=j+1}^{[\alpha'' n]-1} r_k \right) \frac{1}{r_{[\alpha' n]}^{[\alpha' n]-[\alpha'' n]}} \left(\prod_{k=[\alpha'' n]}^{[\alpha' n]-1} r_k \right) \\
&\leqslant \frac{1}{r_{[\alpha' n]}^{[\alpha'' n]-1-j}} \left(\prod_{k=j+1}^{[\alpha'' n]-1} r_k \right) \leqslant \left(\frac{r_{[\alpha'' n]}}{r_{[\alpha' n]}} \right)^{[\alpha'' n]-j-1} \\
&\leqslant \left(\frac{r_{[\alpha'' n]}}{r_{[\alpha' n]}} \right)^{[\alpha'' n]-n-1}.
\end{aligned}$$

In view of (4), an elementary computation gives

$$\limsup_{n\in L} \left(\frac{S_{1,n}}{M(r_{[\alpha' n]})} \right)^{1/n} < 1.$$

(b) By our choice of α we have $E_n = \delta_n^{[\alpha n]} \gamma_{[\alpha n]}$, where $\lim_{n \in L} \delta_n = 0$. Therefore, for all $n+1 \leqslant j \leqslant [\alpha n]$,

$$\begin{aligned} E_{j-1} R^j_{[\alpha' n]} &\leqslant E_n R^{[\alpha n]}_{[\alpha' n]} = 2^{[\alpha n]} E_n r^{[\alpha n]}_{[\alpha' n]} (1 + d^{(1)}_{[\alpha' n]})^{[\alpha n]} \\ &= 2^{[\alpha n]} \delta_n^{[\alpha n]} \gamma_{[\alpha n]} r^{[\alpha n]}_{[\alpha' n]} (1 + d^{(1)}_{[\alpha' n]})^{[\alpha n]} \\ &= M(r_{[\alpha' n]})\, 2^{[\alpha n]} \left\{ \delta_n^{[\alpha n]} \frac{M(r_{[\alpha n]})}{M(r_{[\alpha' n]})} \left(\frac{r_{[\alpha' n]}}{r_{[\alpha n]}} \right)^{[\alpha n]} \right\} (1 + d^{(1)}_{[\alpha' n]})^{[\alpha n]} \\ &\leqslant M(r_{[\alpha' n]})\, 2^{[\alpha n]} \delta_n^{[\alpha n]} (1 + d^{(1)}_{[\alpha' n]})^{[\alpha n]}, \end{aligned}$$

where the last inequality follows by the definition of $r_{[\alpha n]}$. We obtain

$$\limsup_{n \in L} \left(\frac{S_{2,n}}{M(r_{[\alpha' n]})} \right)^{1/n} = 0.$$

(c) For every $j > [\alpha n]$ we have by Lemma 4

$$\begin{aligned} E_{j-1} R^j_{[\alpha' n]} &\leqslant \gamma_j r^j_{[\alpha' n]} (1 + d^{(2)}_j)^j (1 + d^{(1)}_{[\alpha' n]})^j \\ &= M(r_{[\alpha' n]}) \left\{ \frac{M(r_j)}{M(r_{[\alpha' n]})} \frac{r^j_{[\alpha' n]}}{r^j_j} \right\} (1 + d^{(2)}_j)^j (1 + d^{(1)}_{[\alpha' n]})^j \\ &\leqslant M(r_{[\alpha' n]}) \left\{ \frac{r_j^j}{r^{[\alpha' n]+1}_{[\alpha' n]}} \left(\prod_{k=[\alpha' n]+1}^{j-1} \frac{1}{r_k} \right) \frac{r^j_{[\alpha' n]}}{r^j_j} \right\} \\ &\quad \times (1 + d^{(2)}_j)^j (1 + d^{(1)}_{[\alpha' n]})^j \\ &= M(r_{[\alpha' n]}) \left\{ r^{j-1-[\alpha' n]}_{[\alpha' n]} \left(\prod_{k=[\alpha' n]+1}^{j-1} \frac{1}{r_k} \right) \right\} (1 + d^{(2)}_j)^j (1 + d^{(1)}_{[\alpha' n]})^j. \end{aligned}$$

We choose some arbitrary $\alpha'' \in (\alpha', \alpha)$. Lemma 2 yields

$$\begin{aligned} \{ \cdots \} &\leqslant r^{[\alpha'' n] - [\alpha' n] - 1}_{[\alpha' n]} \left(\prod_{k=[\alpha' n]+1}^{[\alpha'' n]-1} \frac{1}{r_k} \right) r^{j-[\alpha'' n]}_{[\alpha' n]} \left(\prod_{k=[\alpha'' n]}^{j-1} \frac{1}{r_k} \right) \\ &\leqslant r^{j-[\alpha'' n]}_{[\alpha' n]} \left(\prod_{k=[\alpha'' n]}^{j-1} \frac{1}{r_k} \right) \leqslant \left(\frac{r_{[\alpha' n]}}{r_{[\alpha'' n]}} \right)^{j-[\alpha'' n]}, \end{aligned}$$

and thus we obtain that

$$\sum_{j=[\alpha n]+1}^{\infty} E_{j-1} R_{[\alpha' n]}^{j} \leqslant M(r_{[\alpha' n]}) \sum_{j=[\alpha n]+1}^{\infty} \left(\frac{r_{[\alpha' n]}}{r_{[\alpha'' n]}}\right)^{j-\alpha'' n} \times (1+d_j^{(2)})^j \, (1+d_{[\alpha' n]}^{(1)})^j$$

$$\leqslant M(r_{[\alpha' n]}) \left(\frac{r_{[\alpha' n]}}{r_{[\alpha'' n]}}\right)^{[\alpha n]-[\alpha'' n]} \sum_{j=1}^{\infty} \left(\frac{r_{[\alpha' n]}}{r_{[\alpha'' n]}}\right)^{j} \times (1+d_{j+[\alpha n]}^{(2)})^{j+[\alpha n]} \, (1+d_{[\alpha' n]}^{(1)})^{j+[\alpha n]}$$

$$=: M(r_{[\alpha' n]}) \left(\frac{r_{[\alpha' n]}}{r_{[\alpha'' n]}}\right)^{[\alpha n]-[\alpha'' n]} S_n.$$

An elementary calculation shows that each series S_n, $n \in \mathbb{N}$, is convergent and that $\lim_{n \to \infty} S_n^{1/n} = 1$. Hence, we have by (4)

$$\limsup_{n \in L} \left(\frac{S_{3,n}}{M(r_{[\alpha' n]})}\right)^{1/n} < 1.$$

Putting (a), (b), and (c) together we obtain that for some $\alpha' \in (1, 1+\delta)$

$$\limsup_{n \in L} \left(\frac{\phi(r_{[\alpha' n]})}{M(r_{[\alpha' n]})}\right)^{1/n} < 1,$$

which contradicts (5).

Proof of Theorem 3. We choose

$$M(r) = \exp(\tau r^{\rho}), \qquad \text{i.e.,} \qquad r_n = \left(\frac{n}{\tau\rho}\right)^{1/\rho}$$

and obtain that for all $1 \leqslant \alpha < \alpha'$

$$\lim_{n \to \infty} \frac{r_{[\alpha' n]}}{r_{[\alpha n]}} = \left(\frac{\alpha'}{\alpha}\right)^{1/\rho} > 1.$$

Further, since f is of perfectly regular growth, we have

$$\tau = \lim_{r \to \infty} \frac{\log \phi(r)}{r^{\rho}} = \lim_{n \to \infty} \frac{\log \phi(r_n)}{r_n^{\rho}} = \tau\rho \lim_{n \to \infty} \log \phi(r_n)^{1/n},$$

which implies

$$\lim_{n \to \infty} \left(\frac{\phi(r_n)}{M(r_n)} \right)^{1/n} = 1.$$

By Theorem 2 it follows that (1) holds for $L = \mathbb{N}$.

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